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THE ASYMPTOTIC BEHAVIOUR OF ATTAINABLE SETS OF SINGULARLY-PERTURBED LINEAR AUTONOMOUS CONTROL SYSTEMS†

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The asymptotic properties of attainable sets of singularly-perturbed linear autonomous control systems are investigated. It is shown that if an explicitly given linear scaling operator is applied to the attainable set, the resulting sets converge (as the small parameter tends to zero). © 1999 Elsevier Science Ltd. All rights reserved.

We consider a singularly-perturbed linear control system with a small parameter ε as a coefficient of the derivatives of the fast components of the state vector, over a finite time interval $t \in [0, T]$, and investigate the asymptotic behaviour of its attainable sets $K(\varepsilon, t)$ as $\varepsilon \to 0$. It has been proved [1] that if the system is stable with respect to the fast variables, then $K(\varepsilon, t)$ converges. For systems without slow variables the convergence has been proved [2] for the shapes of the attainable sets rather than for the attainable sets themselves (by the shape of a set we mean the entity of all its images under non-singular linear transformations).

In the general case considered here, it is possible to indicate a matrix scaling function $R(\varepsilon, t)$ such that the product of this function and the attainable set $K(\varepsilon, t)$ tend to a limit as $\varepsilon \to 0$, describing in this way the asymptotic properties of the attainable sets themselves. In the language of shapes (applicable only to systems such that their attainable sets are bodies), this means that the shapes of the attainable sets $K(\varepsilon, t)$ converge.

1. STATEMENT OF THE PROBLEM

Consider a singularly-perturbed linear control system

$$\dot{x} = Ax + By + Fu, \quad x(0) = y(0) = 0 \tag{1.1}$$

$$\varepsilon\dot{y} = Cx + Dy + Gu, \quad t \in [0, T]$$

where $x \in V_x = \mathbf{R}^n$, $y \in V_y = \mathbf{R}^m$ are the slow and fast components of the state vector, $\varepsilon > 0$ is a small positive parameter, and the admissible control vectors $u(t)$ belong at any time to a convex compact $U \subset \mathbf{R}^k$ containing zero and are measurable functions of time. We assume that D is a non-singular matrix (which is essential) such that its pure imaginary eigenvalues are multiplicity free (this simplifies the form of the matrix $F(\varepsilon, t)$).

Let $K(\varepsilon, t)$ be an attainable set of system (1), that is, the set of all points of the space $V = V_x \oplus V_y$ that the system can reach under an admissible control $u(\tau)$, $\tau \in [0, t]$. For any ε, t, the set $K(\varepsilon, t)$ is a convex compact containing zero.

Let S_0 and S_1 be sets in V. The Hausdorff distance between them is defined by

$$\rho(S_0, S_1) = \inf\{r: \forall s_i \in S_i \; \exists s_{1-i} \in S_{1-i}: |s_0 - s_1| < r, \; i = 0, 1\}$$

We will investigate the asymptotic behaviour $K(\varepsilon, t)$ with respect to the Hausdorff metric as $\varepsilon \to 0$.

2. SEPARATION OF THE FAST AND SLOW VARIABLES

We make the following change of variables in system (1.1)

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$$\left\|\begin{matrix}x\\y\end{matrix}\right\| = T_0\left\|\begin{matrix}x\\z\end{matrix}\right\|, \quad T_0 = \left\|\begin{matrix}I & 0\\-D^{-1}C & I\end{matrix}\right\|$$

(Throughout, I stands either for the unit matrix of appropriate size or the corresponding identity operator.) This change of variables corresponds to the representation of the space as $V = V_x \oplus V_z$. It brings system (1.1) to the form

$$\left\|\begin{matrix}\dot{x}\\\dot{z}\end{matrix}\right\| = \left\|\begin{matrix}A_1 & B\\C_1 & \varepsilon^{-1}D_1(\varepsilon)\end{matrix}\right\|\left\|\begin{matrix}x\\z\end{matrix}\right\| + \left\|\begin{matrix}F\\\varepsilon^{-1}G_1(\varepsilon)\end{matrix}\right\|u, \quad x(0) = z(0) = 0 \tag{2.1}$$

$$A_1 = A - BD^{-1}C,\; C_1 = D^{-1}C(A - BD^{-1}C),\; D_1(\varepsilon) = D + D^{-1}CB\varepsilon,\; G_1(\varepsilon) = G + D^{-1}CF\varepsilon.$$

The coordinate transformation T_0 brings the matrix of the system from the form

$$\left\|\begin{matrix}A & B\\\varepsilon^{-1}C & \varepsilon^{-1}D\end{matrix}\right\|$$

to

$$\left\|\begin{matrix}A_1 & B\\C_1 & \varepsilon^{-1}D_1(\varepsilon)\end{matrix}\right\|$$

and increases the order of the off-diagonal block with respect to ε. Similar transformations can bring the matrix of the system to a form where the off-diagonal blocks are small to arbitrarily large order with respect to ε.

Indeed, suppose that in some coordinate system the matrix of the system takes the form

$$\left\|\begin{matrix}A(\varepsilon) & \varepsilon^{l}B(\varepsilon)\\\varepsilon^{k}C(\varepsilon) & \varepsilon^{-1}D(\varepsilon)\end{matrix}\right\| \tag{2.2}$$

where A, B, C, D are matrix polynomials with respect to ε, $D(0)$ is invertible and $k \geqslant 0$, $l \geqslant 0$. Consider the coordinate transformations with matrices

$$\left\|\begin{matrix}I & 0\\\varepsilon^{k+1}Y & I\end{matrix}\right\|, \quad \left\|\begin{matrix}I & \varepsilon^{l+1}X\\0 & I\end{matrix}\right\|; \quad X = B(0)D^{-1}(0), \quad Y = -D^{-1}(0)C(0)$$

Under these transformations, the matrix of system (2.2) takes a form similar to (2.2) with k replaced by $k + 1$ under the first transformation and l replaced by $l + 1$ under the second; all the functions remain polynomial in ε, and the principal terms of the expansions of the off-diagonal blocks with respect to ε, and the principal terms of the expansions of the off-diagonal blocks with respect to ε are preserved.

It is known [3] that, for all sufficiently small ε, a coordinate transformation $S(\varepsilon)$

$$\left\|\begin{matrix}x\\z\end{matrix}\right\| = S(\varepsilon)\left\|\begin{matrix}p\\q\end{matrix}\right\|$$

exists which is close to the identity and reduces the matrix of system (2) to a matrix with zero off-diagonal blocks. It can be shown that $S(\varepsilon)$ is an analytic function and can be expressed as

$$S(\varepsilon) = \left\|\begin{matrix}I & \varepsilon X(\varepsilon)\\\varepsilon Y(\varepsilon) & I\end{matrix}\right\|, \quad X(0) = BD^{-1}, \quad Y(0) = -D^{-1}C_1$$

Thus, the coordinate transformation $S(\varepsilon)$ brings system (2) to the form

$$\left\|\begin{matrix}\dot{p}\\\dot{q}\end{matrix}\right\| = \left\|\begin{matrix}A(\varepsilon) & 0\\0 & \varepsilon^{-1}D(\varepsilon)\end{matrix}\right\|\left\|\begin{matrix}p\\q\end{matrix}\right\| + \left\|\begin{matrix}F(\varepsilon)\\\varepsilon^{-1}G_2(\varepsilon)\end{matrix}\right\|u, \quad p(0) = q(0) \tag{2.3}$$

where all functions are analytic, $A(0) = A_1$, $D(0) = D$, $F(0) = F - BD^{-1}G$, $G_2(0)$. This corresponds to a representation of the phase space as $V = Vp \oplus V_q$, such that $V_p \to V_x$, and $V_q \to V_z$ as $\varepsilon \to 0$. Thus, a suitable choice of a coordinate system depending on ε separates the slow and fast variables in the dynamics of the system, and the only connection between the two kinds of variables is now via the control.

3. DECOMPOSITION OF THE OPERATORS D AND $D(\varepsilon)$ INTO UNSTABLE, NEUTRAL, AND STABLE COMPONENTS, AND THE CORRESPONDING DECOMPOSITION OF THE SPACES OF THE FAST VARIABLES V_z AND V_q

Consider the operator in the space V_z defined by the matrix D in z coordinates. Consider the decomposition of the space V_z.

$$V_z = V_z^+ \oplus V_z^0 \oplus V_z^-$$

corresponding to the decomposition of D into the direct sum of unstable, neutral and stable operators in accordance with the signs of the real parts of their eigenvalues

$$D = D_+ \oplus D_0 \oplus D_-$$

$$\lambda(D_i) = \lambda_i, \quad i = +, 0, -; \quad \operatorname{Re}\lambda_+ > 0, \quad \operatorname{Re}\lambda_0 = 0, \quad \operatorname{Re}\lambda_- < 0$$

In the spaces V_z^i introduce the coordinates z_i. We may assume without loss of generality, that $z = \operatorname{col}(z_+, z_0, z_-)$. This assumption holds if the matrix D is block-diagonal with the matrices of the operators D_i along the diagonal. This is always the case on applying the transformation $y = H\tilde{y}$ in system (1).

The eigenvalues $\lambda(\varepsilon)$ of the operator $D(\varepsilon)$ in the space V_q are continuous functions of ε. We divide them into three families, according to their limiting values: $\lambda_+(\varepsilon) \to \lambda_+$, $\lambda_0(\varepsilon) \to \lambda_0$, $\lambda_-(\varepsilon) \to \lambda_-$. Consider the representation of the space V_q

$$V_q = V_r^+ \oplus V_r^0 \oplus V_r^-$$

corresponding to the direct-sum decomposition of the operator $D(\varepsilon)$ as

$$D(\varepsilon) = D_+(\varepsilon) \oplus D_0(\varepsilon) \oplus D_-(\varepsilon), \quad \lambda(D_i(\varepsilon)) = \lambda_i(\varepsilon), \quad i = +, 0, -$$

Since the matrices D and $D(\varepsilon)$, as well as the spaces V_z and V_q, are close to each other when ε is small, it follows that the same is true for the subspaces, that is

$$V_r^i \to V_z^i, \ i = +, \ 0, \ -$$

as $\varepsilon \to 0$. In the spaces V_r^i we introduce coordinates r_i close to z_i, and in V_q we introduce coordinates $r = \operatorname{col}(r_+, r_0, r_-)$. We use the variables $\operatorname{col}(p, r)$ in Eqs (2.3). The matrix $S_1(\varepsilon)$ of the coordinate transformation

$$\begin{Vmatrix} p \\ q \end{Vmatrix} = S_1(\varepsilon) \begin{Vmatrix} p \\ r \end{Vmatrix}$$

is analytic and close to the unit matrix. Equation (2.3) takes the form

$$\operatorname{col}(\dot{p}, \dot{r}_+, \dot{r}_0, \dot{r}_-) = \operatorname{diag}(A(\varepsilon), \varepsilon^{-1}D_+(\varepsilon), \varepsilon^{-1}D_0(\varepsilon), \varepsilon^{-1}D_-(\varepsilon))\operatorname{col}(p, r_+, r_0, r_-)$$
$$+\operatorname{col}(F(\varepsilon), \varepsilon^{-1}G(\varepsilon))u \qquad (3.1)$$

The function $G(\varepsilon)$ is analytic and $G(0) = G$. Thus, the variables in Eq. (3.1) are divided into slow, fast unstable, fast neutral and fast stable variables, interconnected only via the control. They are related to the variables x, z by an analytic matrix close to the unit matrix.

4. INTRODUCTION OF THE SCALING OPERATOR. CONVERGENCE OF SETS RELATED TO THE ATTAINABLE SETS

We introduce a scaling operator $R(\varepsilon, t)$ as the direct sum of operators in the corresponding subspaces with respect to the space representation

$$V = V_p \oplus V_r^+ \oplus V_r^0 \oplus V_r^-$$

namely

$$R(\varepsilon, t) = R_p \oplus R_r^+ \oplus R_r^0 \oplus R_r^-$$

$$R_p = I, \quad R_r^+ = \exp(-\varepsilon^{-1} D_+(\varepsilon)t), \quad R_r^0 = \varepsilon I, \quad R_r^- = I$$

(In general, if the eigenvalue $i\omega_k$ of the operator D_0 corresponds to a Jordan's block of size m_k ($k = 1, \ldots, N$), then the matrix of the operator $R^0{}_r$ must have the form

$$R_r^0 = \operatorname{diag}(R_1, \ldots, R_N), \quad R_k = \operatorname{diag}(\varepsilon, \varepsilon^2, \ldots, \varepsilon^{m_k}), \quad k = 1, \ldots, N$$

with respect to the basis, where the matrix of the operator D_0 has the Jordan normal form. It can be proved that all results stated below remain true for such an operator R_r^0.)

Apply the scaling operator to the attainable set and consider the set $Q(\varepsilon, t) = R(\varepsilon, t)K(\varepsilon, t)$ obtained. The set $Q(\varepsilon, t)$ is the range of the vector

$$\operatorname{col}(p, s) = R(\varepsilon, t) \operatorname{col}(p, r)$$

$\operatorname{col}(p, s) = \operatorname{col}(p, s_+, s_0, s_-) = \operatorname{diag}(I, \exp(-\varepsilon^{-1} D_+(\varepsilon)t, \varepsilon I, I)$. To describe the asymptotic behaviour of $K(\varepsilon, t)$ we investigate that of the sets $Q(\varepsilon, t)$. We will prove that the sets $Q(\varepsilon, t)$ converge with respect to the Hausdorff metric as $\varepsilon \to 0$, for any $t \in [0, T]$.

We need some more notation. Let $V^0 = V_r^0 \oplus Vp$, be the direct sum of the spaces of the slow and fast neutral variables. Denote by $(\cdot)_+, (\cdot)_0, (\cdot)_-$ the projections onto the subspaces V_r^+, V^0, V_r^-, respectively, and denote by Π and Π^0 the projections onto V_p and V_r^0 (both spaces and operators depend analytically on ε, though this dependence is not made explicit in the notation, for brevity).

Theorem 1. For any $t \in [0, T]$ limits exist as $\varepsilon \to 0$ of the projections $Q(\varepsilon, t)$ onto the spaces V_r^+, V_r^- and $V^0 = V_r^0 \oplus Vp$

$$\lim_{\varepsilon \to 0} Q_i(\varepsilon, t) = Q_i \subset V_z^i, \quad i = +, -; \quad \lim_{\varepsilon \to 0} Q_0(\varepsilon, t) = Q_0(t) \subset V_z^0 \oplus V_x$$

Proof. We begin with $Q(\varepsilon, t)_-$. We have

$$\operatorname{col}(p, s)_- = s_- = r_- = \int_0^{t/\varepsilon} \exp\left(D_-(\varepsilon)\left(\frac{t}{\varepsilon} - s\right)\right)(G(\varepsilon)u(\varepsilon s))_- ds \tag{4.1}$$

$$Q_-(\varepsilon, t) = \int_0^{t/\varepsilon} \exp(D_-(\varepsilon)s)(G(\varepsilon)U)_- ds$$

For sufficiently small ε, we have $\operatorname{Re} \lambda_-(\varepsilon) \leq \tilde{\lambda} < 0$, whilst the set U of control vectors is bounded; hence $|\exp(D_-(\varepsilon)s)(G(\varepsilon)U)_-| <$ const $\exp(\lambda s)$. Since the improper integral of the majorizing function is convergent, and for all s

$$\exp(D_-(\varepsilon)s)(G(\varepsilon)U)_- \to \exp(D_- s)(GU)_- \text{ as } \varepsilon \to 0$$

it follows by Lebesgue's theorem that

$$\lim_{\varepsilon \to 0} Q_-(\varepsilon, t) = \int_0^\infty \exp(D_- s)(GU)_- ds = Q_- \subset V_z^- \tag{4.2}$$

A similar proof yields the convergence of $Q(\varepsilon, t)_+$. We have

$$\operatorname{col}(p, s)_+ = s_+ = \exp(-\varepsilon^{-1} D_+(\varepsilon)t) r_+ = \int_0^{t/\varepsilon} \exp(-D_+(\varepsilon)s)(G(\varepsilon)u(\varepsilon s))_+ ds \tag{4.3}$$

so that

$$\lim_{\varepsilon \to 0} Q_+(\varepsilon, t) = \int_0^\infty \exp(-D_+ s)(GU)_+ ds = Q_+ \subset V_z^+ \tag{4.4}$$

Finally, we consider the projection $(Q(\varepsilon, t))_0$. The set $(K(\varepsilon, t))_0$ is the range of the vector $\operatorname{col}(p(\varepsilon, t), s_0(\varepsilon, t))$, where

$$p(\varepsilon, t) = \int_0^t \exp(A(\varepsilon)s)F(\varepsilon)u(t-s)ds$$

$$s_0(\varepsilon, t) = \int_0^t \exp(\varepsilon^{-1}D_0(\varepsilon)s)\Pi_0 G(\varepsilon)u(t-s)ds \tag{4.5}$$

Let h and $H_{\varepsilon,t}$ the support functions of the sets U and $(Q(\varepsilon, t))_0$. Fix a vector ξ in V and consider the corresponding value of $H_{\varepsilon,t}$

$$H_{\varepsilon,t}(\xi) = \sup_{(p,s_0)\in Q_0} (p^*\Pi\xi + s_0^*\Pi_0\xi) =$$

$$= \int_0^t \sup_u (u^*F^*(\varepsilon)\exp(A^*(\varepsilon)s)\Pi\xi + u^*G^*(\varepsilon)\Pi_0^*\exp(\varepsilon^{-1}D_0^*(\varepsilon)s)\Pi_0\xi)ds =$$

$$= \varepsilon\int_0^{t/\varepsilon} h(F^*(\varepsilon)\exp(A^*(\varepsilon)\varepsilon s)\Pi\xi + G^*(\varepsilon)\Pi_0^*\exp(D_0^*(\varepsilon)s)\Pi_0\xi)ds \tag{4.6}$$

(the asterisk stands for transposition). We may assume that the matrix $D_0^*(\varepsilon)$ has the Jordan normal form (with respect to a suitable basis of the space V_r^0). Since the eigenvalues of D_0 are distinct, the same is true for $D_0^*(\varepsilon)$, so the matrix $D_0^*(\varepsilon)$ is diagonal. Its eigenvalues have the form

$$\lambda_k^0(\varepsilon) = \alpha_k(\varepsilon) + i\omega_k(\varepsilon) = \varepsilon\alpha_k^1 + \varepsilon^2\alpha_k^2 + \ldots + i(\omega_k + \varepsilon\omega_k^1 + \varepsilon^2\omega_k^2 + \ldots),\ k = 1,\ldots,N$$

Therefore, the matrix elements of $\exp(D_0^*(\varepsilon)s)$ have the form

$$\exp(\varepsilon s\alpha_k^1 + \varepsilon^2 s\alpha_k^2 + \ldots)\exp(i\omega_k s)\exp(i(\varepsilon s\omega_k^1 + \varepsilon^2 s\omega_k^2 + \ldots)),\ k = 1,\ldots,N$$

Hence we may treat the term $h(\cdot)$ in (10) as a continuous function f of the variables $\omega_1 s, \ldots, \omega_N s$, εs, ε, which is 2π-periodic with respect to the first N arguments. It can be approximated to within any accuracy by a finite trigonometric polynomial

$$f(\omega_1 s, \ldots \omega_N s, \varepsilon s, \varepsilon) \sim \sum_{m\in Z^N} a_m(\varepsilon s, \varepsilon)\exp(is(m, \omega))$$

Then

$$H_{\varepsilon,t}(\xi) = \varepsilon\int_0^{t/\varepsilon} f(\omega_1 s, \ldots \omega_N s, \varepsilon s, \varepsilon)ds \sim \sum_{m\in Z^N} \varepsilon\int_0^{t/\varepsilon} 0a_m(\varepsilon s, \varepsilon)\exp(is(m, \omega))ds$$

If m is such that $(m, \omega) = 0$ we have

$$\int_0^{t/\varepsilon} a_m(\varepsilon s, \varepsilon)\exp(is(m, \omega))ds = \int_0^1 a_m(t, \varepsilon)dt \to \int_0^1 a_m(t, 0)dt \text{ as } \varepsilon \to 0$$

If m is such that $(m, \omega) \neq 0$ we have

$$\varepsilon\int_0^{t/\varepsilon} a_m(\varepsilon s, \varepsilon)\exp(is\alpha)ds = \int_0^1 a_m(t, \varepsilon)\exp(i\alpha t/\varepsilon)dt \to 0 \text{ as } \varepsilon \to 0$$

by the Riemann–Lebesgue lemma. Thus, we see that for any ξ

$$H_{\varepsilon,t}(\xi) \to \int_0^1 \sum_{m\in Z^N,(m,\omega)=0} a_m(t, 0)dt \quad \text{as} \quad \varepsilon \to 0 \tag{4.7}$$

Assertion. Convergence with respect to the Hausdorff metrices of a sequence of convex compacts S_n is equivalent to the pointwise convergence and uniform boundedness of their support functions $H_n(\xi)$ on the sphere $|\xi| = 1$.

Proof. The sequence S_n is convergent if and only if the sequence of support functions $H_n(\xi)$ is uniformly convergent on $|\xi| = 1$. This follows from the fact that $\rho(S_n, S) = \max|H_n(\xi) - H(\xi)|$ [4]. Moreover, it is known that the support functions are uniformly convergent if and only if they are pointwise convergent and uniformly bounded.

Since the support functions of the convex sets $(Q(\varepsilon, t))_0$ are pointwise convergent and uniformly bounded (as is evident from (10)), it follows from the above assertion that the limit of the sets themselves does exist.

$$\lim_{\varepsilon \to 0} Q_0(\varepsilon, t) = Q_0(t) \subset V_z^0 \oplus V_x$$

The theorem is proved.

Theorem 2. For any $t \in [0, T]$ the set $Q(\varepsilon, t)$ converges as $\varepsilon \to 0$ to the direct product of its projections onto the spaces V_r^+, V_r^- and $V^0 = V_r^0 \oplus Vp$

$$Q(\varepsilon, t) \to Q_+(\varepsilon, t) \oplus Q_0(\varepsilon, t) \oplus Q_-(\varepsilon, t) \text{ as } \varepsilon \to 0$$

Proof. It is obvious from (4.3), (4.5) and (4.1) that the controls substantially affect the projections $\mathrm{col}(p, s)_i$, $i = +, 0$—only at times close to zero, intermediate, and close to t, respectively. In a time $[0, \sqrt{(t\varepsilon)}]$ system (3.1) may be steered to a point whose projection onto V_r^+ is close to any point of $(Q(\varepsilon, t))_+$, then in a time $[\sqrt{(t\varepsilon)}, t - \sqrt{(t\varepsilon)}]$ it may be steered to a point whose projection onto V^0 is close to any point of $(Q(\varepsilon, t))_0$, affecting the first projection only slightly at the same time and then at a time $[t - \sqrt{(t\varepsilon)}, t]$ it may be steered to a point whose projection onto V_r^- is close to any point $(Q(\varepsilon, t))_-$, changing the first two projections only slightly at the same time. But this also means that the set $Q(\varepsilon, t)$ is close to the direct product of its projections onto V_r^+, V^0 and V_r^-. We also note that $p(\varepsilon, t)$ and $(s(\varepsilon, t))_0$ depend on the control over the same time interval, so that $(Q(\varepsilon, t))_0$ is not close to the direct product of its projections $\Pi K(\varepsilon, t)$ and $\Pi_0 K(\varepsilon, t)$.

Remark. Since $K(\varepsilon, t) = R^{-1}(\varepsilon, t)K(\varepsilon, t)$, it follows from Theorem 2 that $K(\varepsilon, t)$ can be approximated by a direct product of subsets of the spaces V_r^+ V^0 and V_r^- where the first increases exponentially, the third is constant, and the second increases along the space V_r^0 at a rate $1/\varepsilon$

$$K(\varepsilon, t) \sim \exp(\varepsilon^{-1} D_+(\varepsilon) t) Q_+(\varepsilon, t) \oplus (I \oplus \varepsilon^{-1} I) Q_0(\varepsilon, t) \oplus Q_-(\varepsilon, t)$$

The left- and right-hand sides of this equivalence are close together in the Banach–Mazur metric, to be defined below.

Theorems 1 and 2 directly imply the following.

Theorem 3. For any $t \in [0, T]$, the sets $Q(\varepsilon, t)$ tend to a limit which is the direct product of the subsets Q_+, $Q_0(t)$, Q_- of the spaces V_z^+, $V_z^0 \oplus V_x$, V_z^-

$$\lim_{\varepsilon \to 0} Q(\varepsilon, t) = Q(t) = Q_+ \oplus Q_0(t) \oplus Q_-$$

Theorem 3 implies our main result.

Theorem 4. Given a linear singularly-perturbed autonomous control system (1), the sets obtained by applying the linear operators $R(\varepsilon, t)$ defined above to the attainable sets $K(\varepsilon, t)$ tend to a limit as $\varepsilon \to 0$ for any $t \in [0, T]$

$$\lim_{\varepsilon \to 0} R(\varepsilon, t) K(\varepsilon, t) = Q(t) = Q_+ \oplus Q_0(t) \oplus Q_-$$

and the limit set is uniquely determined by its projections given by formulae (4.2), (4.4) and (4.7).

Let us restate this result in the language of shapes of sets. By the shape $\bar{\Omega}$ of a set Ω we mean the entity of its images under non-singular linear transformations $\bar{\Omega} = \{G\Omega : \det G \neq 0\}$. For sets containing a full-dimensional neighbourhood of zero (bodies), one can define the Banach–Mazur distance between them

$$\rho(\Omega_1, \Omega_2) = \log(g(\Omega_1, \Omega_2) g(\Omega_2, \Omega_1)), \quad g(\Omega_1, \Omega_2) = \inf\{g \geq 1; g\Omega_1 \supset \Omega_2\}$$

and the distance between their forms

$$\rho(\bar{\Omega}_1, \bar{\Omega}_2) = \inf_{\det G \neq 0} \rho(G\Omega_1, \Omega_2)$$

This enables us to speak of the convergence of shapes. Assume that the pairs (A_1, F_1) and (D, G) are completely controllable. Then [5] system (1.1) is completely controllable for all sufficiently small ε if the bounds on the control are dropped. Consequently, the sets $K(\varepsilon, t)$ are bodies. Since the shapes of the sets $K(\varepsilon, t)$ and $R(\varepsilon, t)K(\varepsilon, t)$ are identical, the result of Theorem 4 may be restated in the language of shapes of attainable sets as follows.

Theorem 5. The shapes of the attainable sets of a singularly-perturbed autonomous control system (1) tend to a limit as $\varepsilon \to 0$ for any $t \in [0, T]$

$$\lim_{\varepsilon \to 0} \overline{K}(\varepsilon, t) = \overline{K}(t).$$

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